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Core Solutions in Vector-Valued Games[1]

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Abstract. In this paper, we analyze core solution concepts for vector-valued cooperative games. In these games, the worth of a coalition is given by a vector rather than by a scalar. Thus, the classical concepts in cooperative game theory have to be revisited and redefined; the important principles of individual and collective rationality must be accommodated; moreover, the sense given to the domination relationship gives rise to two different theories. Although different, we show the areas which they share. This analysis permits us to propose a common solution concept that is analogous to the core for scalar cooperative games.

Key Words. Game theory, multicriteria games, solution concepts, core.

1. Introduction

Nowadays, the theory of games with vector-valued payoffs has experienced an important development. Since the seminal papers by Blackwell (Ref. 1) and Shapley (Ref. 2), many papers have been published in both noncooperative and cooperative theories.

The vector-valued cooperative game theory is a realistic way to model conflict situations because it incorporates into the analysis all the criteria that should be considered in the problem. Most of the times, the agents involved in decision-making models must consider simultaneously several aspects in the negotiation and these aspects cannot be isolated. Cooperative vector-valued games arise naturally when modeling cooperation between

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agents that consider simultaneously different aspects in the negotiation. For example, distribution companies such as TV networks and cell-phone networks face this type of decision when they want to enter a new market. These companies have to combine at least two aspects in the negotiation: profit in the short run and coverage of the market, to improve their positions in the medium and long run. Analyzing each aspect of the negotiation separately may lead to unresolved situations: one may agree to some aspects of cooperation but disagree to others.

There is another important reason to deal with vector-valued games. Any model with scalar payoffs has its immediate counterpart as soon as uncertainty in the estimation of such payoff is considered. This approach gives rise to the scenario analysis. In a scenario analysis, we are given several instances of the payoff function, each representing an admissible realization of our true function. No probabilistic assumptions are known about these functions (scenarios), and the question is to search for compromise solutions that are good simultaneously in all of the scenarios where the problem can occur.

Example 1.1. Consider three cell-phone operators (namely O_1, O_2, and O_3) that want to enter a new market. There are two criteria that must be considered in the process. On the one hand, there is the profit that has been estimated from the market analysis. On the other hand, there is the coverage, which is regulated by law. Thus, the percentage of population covered by each operator or by merging is fixed by the government. Coverage is very important because it is known to improve the return in the medium and long run. Let us assume that profit is measured in millions of dollars and coverage in percent. We represent by vectors with two entries the values obtained by each operator: the first entry is the profit and the second one is the coverage. Let us consider the following data that represent the values obtained in different cooperation situations:

S	$\{O_1\}$	$\{O_2\}$	$\{O_3\}$	$\{O_1, O_2\}$	$\{O_1, O_3\}$	$\{O_2, O_3\}$	$\{O_1, O_2, O_3\}$
$v(S)$	$\begin{bmatrix}2\\20\end{bmatrix}$	$\begin{bmatrix}3\\40\end{bmatrix}$	$\begin{bmatrix}4\\10\end{bmatrix}$	$\begin{bmatrix}6\\70\end{bmatrix}$	$\begin{bmatrix}6\\30\end{bmatrix}$	$\begin{bmatrix}11\\40\end{bmatrix}$	$\begin{bmatrix}12\\100\end{bmatrix}$.

It is clear that we cannot analyze separately the profit game and the coverage game. For the profit game, the core is empty. On the other hand, for the coverage game, clearly $y = (30, 40, 30)$ is a core allocation. Therefore, it is not easy to answer the question of whether the operators should cooperate to enter the market. □

It is clear that there exists a need to develop a unified theory to help in this analysis and to fill in the existing gap. When dealing with several criteria, in the classical analysis one may try two different strategies: (i) separation of a game into its component games; (ii) combination of the criteria into a utility function by means of a vector of weights λ. It is clear that the first approach is too simplistic and results in the loss of insight in these games; see for instance the previous example. On the other hand, several authors have suggested converting vector-valued games into weighted scalar games; see Refs. 1 and 3–5. In Refs. 1, 3, 4, the weights represent a distribution of probability on the criteria; consequently, the weighted payoff represents the expected payoff. In Ref. 5, there is another interpretation of weights as tradeoffs between the criteria.

Now, we consider the class of vector-valued transferable utility games (TU games), that is, vector-valued games where the players accept side payments. We analyze this class of games from a multicriteria perspective. Similar analyses for noncooperative games have been considered also in Refs. 6–9. In this paper, we extend the classical individual and collective rationality principles using two different orderings in the payoff space. The first one corresponds to a compromise attitude toward negotiation where coalitions admit payoffs that are not worse in all the components than what they can ensure by themselves. The second one is a more restrictive ordering which accepts only payoffs that get more in all of the components.

Partial approaches to these two analyses have been done in Refs. 5 and 10 and an application can be seen in Ref. 11. Although it seems that these two approaches lead to two different theories, we show that they share a common solution concept, which we recommend as solution for this family of games: the set of nondominated imputations by allocations (NDIA). The main properties of NDIA are studied and its relationship with the classical concept of core is shown.

The paper is organized as follows. In Section 2, we introduce the definition of vector-valued cooperative game and the concept of allocation for these games. Moreover, we analyze two different domination relationships that extend the classic domination concept in the scalar case. Section 3 analyzes the solution concepts for vector-valued cooperative games using a compromise ordering: players accept side payments that are not worse componentwise than their guarantee payoffs. Section 4 studies the solution concepts under strong ordering. In this case, the players accept side payments provided that they get more in all of the components of their payoff functions. Finally, in the conclusions (Section 5), we recommend a common solution concept valid for the two types of analysis of vector-valued cooperative games.

2. Basic Concepts

A vector-valued cooperative game (N, v) is a set of players $N = \{1, 2, \ldots, n\}$ and a map $v: \mathcal{N} \cup \varnothing \rightarrow \mathbb{R}^m$, such that $v(\varnothing) = 0$, that gives the worth of each coalition by means of different criteria. The elements of the set N are called players and the function v is the characteristic function of the game. A subset S of the player set is called a coalition and $v(s)$ is the worth of coalition S in the game. We denote by G^v the family of all the vector-valued cooperative games and by g^v the family of all the scalar cooperative games.

If a vector-valued game is played and if all the players in N decide to cooperate, then an interesting question which arises is how the vector $v(N)$ should be allocated among the various players.

It is worth noting that this is the same situation which appears in scalar cooperative games, where the worth of $v(N) \in \mathbb{R}$ has to be allocated among the players.

The natural extension of the idea of allocation used in scalar games to vector-valued games consists of using a payoff matrix (an element of $\mathbb{R}^{m \times n}$), whose rows are allocations of the criteria. Since the payoffs are vectors, the allocations in these games are matrices with m rows (criteria) and n columns (players),

$$X = \begin{bmatrix} x_1^1 & x_1^2 & \ldots & x_1^n \\ x_2^1 & x_2^2 & \ldots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ x_m^1 & x_m^2 & \ldots & x_m^n \end{bmatrix}.$$

The ith column X^i in the matrix X represents the payoffs of the ith player for each criteria; therefore,

$$X^i = (x_1^i, x_2^i, \ldots, x_m^i)^t$$

are the payoffs for player i. The jth row X_j in the matrix X is an allocation among the players of the total amount obtained in each criteria;

$$X_j = (x_j^1, x_j^2, \ldots, x_j^n)$$

are the payoffs corresponding to criteria j for each player. The sum

$$X^S = \sum_{i \in S} X^i$$

is the overall payoff obtained by coalition S. The set of allocations of a game (N, v) is denoted by $I^*(N, v)$.

Among all the allocations of the game $(N, v) \in G^v$, we are interested in those which cannot be beaten by the worth given to the coalitions. In scalar games, to beat an allocation with respect to a coalition means to find another allocation which gives more worth to the members of that coalition (equivalently, which gives no less worth). Nevertheless, in vector-valued games, to get more worth is not equivalent to not to get less worth.

These two ways of analyzing the situation correspond to two different solution concepts for these games. In the first one, we do not admit less worth componentwise than what we can guarantee already to ourselves. This would lead us to get more. In the second one, we accept compromise payoffs which get worse in some of the criteria, provided that we increase payoffs in some others.

In the following example, we show allocations which fulfill each one of the above mentioned requirements.

Example 2.1. Consider a three players game $N=\{1,2,3\}$ and two objectives. The payoffs that each player or coalition can obtain by itself are the following:

S	$\{P_1\}$	$\{P_2\}$	$\{P_3\}$	$\{P_1,P_2\}$	$\{P_1,P_3\}$	$\{P_2,P_3\}$	$\{P_1,P_2,P_3\}$
$v(S)$	$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}3\\4\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}6\\7\end{bmatrix}$	$\begin{bmatrix}6\\3\end{bmatrix}$	$\begin{bmatrix}11\\4\end{bmatrix}$	$\begin{bmatrix}12\\10\end{bmatrix}$.

In this example, the matrix

$$X=\begin{bmatrix}3 & 4 & 5\\3 & 5 & 2\end{bmatrix}$$

is an allocation of 12 and 10 in which each player obtains for each criteria at least the payoff that he can ensure for himself [we denote this by $\geqq$]:

$$X^1=(3,3)^t \geqq v(\{1\}),$$

$$X^2=(4,5)^t \geqq v(\{2\}),$$

$$X^3=(5,2)^t \geqq v(\{3\})],$$

and also the matrix X collectively allocates among each coalition more than what each one of them can guarantee for themselves:

$$X^{\{1,2\}}=(7,8)^t \geqq v(\{1,2\}),$$

$$X^{\{1,3\}}=(8,5)^t \geqq v(\{1,3\}),$$

$$X^{\{2,3\}}=(9,7)^t \geqq v(\{2,3\}).$$

However, the matrix

$$Y = \begin{bmatrix} 3 & 5 & 4 \\ 3 & 3 & 4 \end{bmatrix}$$

is another allocation of (12, 10), so that none of the players would refuse it because it is not worse than what they could attain for themselves. In this case, the payoffs for player 2 and for coalition $S = \{1, 2\}$ are not better than the worth they receive by the characteristic function

$$Y^2 = (5, 3)^t \ngeqq v(\{2\}),$$
$$Y^{\{1,2\}} = (8, 6)^t \ngeqq v(\{1, 2\}). \qquad \square$$

Our previous discussion has shown that at least two different orderings are possible in the set of allocations in vector-valued games. One of them is a complete binary relation, although it is not transitive: an allocation a is not worse than another allocation b if a gets more worth than b in at least one criterion. The second one is a partial order: an allocation a is preferred to another allocation b if a gets more worth than b in all the criteria.

If we denote by $\mathbb{R}^m_{\geqq}$ the nonnegative orthant of the m-dimensional real space $\mathbb{R}^m$, it is straightforward to see that a is not worse than b if and only if there exists $\lambda \in \Lambda^m_{\geqq}$ such that

$$\lambda^t a > \lambda^t b$$

and a gets more worth than b in all the criteria if and only if

$$\lambda^t a \geqq \lambda^t b, \qquad \text{for all } \lambda \in \Lambda^m_{\geqq},$$

where (see Ref. 12)

$$\Lambda^m_{\geqq} = \left\{ \lambda \in \mathbb{R}^m_{\geqq} : \sum_{j=1}^{m} \lambda_j = 1 \right\}.$$

Notice that λ_j can be seen as the importance or weight factor assigned to the jth criterion of the payoff, and $\Lambda^m_{\geqq}$ is the whole set of admissible weighting factors. This means that, if a vector of weights is accepted, the players have complete information about the tradeoff among the criteria, and therefore this leads to a scalar game. On the opposite situation, the players may have no information about the tradeoffs. Thus, any weighting coefficient $\lambda \in \Lambda^m_{\geqq}$ may be considered. However, the most realistic situation is when the players have some information on the criteria. It is well known that any ordering can be refined when some additional information on the criteria is available. A natural way of providing additional information is by reducing the set of admissible weights. Some ways of giving additional information, already considered in the literature, are the interval criterion weights (see Refs. 13–15) or linear relations among weights which state interpreference criteria (Refs. 16–17).

From now on, we assume that we are given a closed polyhedron $\Lambda \subseteq \Lambda^m_{\geqq}$ which represents the preferences on the criteria. When we consider the modified set of admissible weights, the sense of the preference ordering changes and only the weights in the set Λ must be considered.

Let $a, b \in \mathbb{R}^m$ be two vectors, we say that:

(i) a is not worse than b according to the information set Λ ($a \overset{\Lambda}{\gtrsim} b$) if and only if

$$\exists \lambda \in \Lambda \text{ such that } \lambda^t a > \lambda^t b; \tag{1}$$

(ii) a is at least as preferred as b according to the information set Λ ($a \overset{\Lambda}{\geqq} b$) if an only if

$$\lambda^t a \geq \lambda^t b, \qquad \forall \lambda \in \Lambda; \tag{2}$$

(iii) a is equivalent to b according to the information set Λ ($a \overset{\Lambda}{\equiv} b$) if and only if

$$\lambda^t a = \lambda^t b, \qquad \forall \lambda \in \Lambda. \tag{3}$$

Example 2.1 (Continued). Let us consider that the set of admissible weights is

$$\Lambda = \{\lambda \in \Lambda^m_{\geqq} : 1/3 \leq \lambda_1 \leq 2/3\}.$$

Using this information set about the weights, none of the players or coalitions can disagree with either the allocation X or the allocation Y. The payoffs for every player or coalition are preferred according to Λ to the payoffs that they could guarantee. In Fig. 1, we show the preference cone, the characteristic function values, and the allocation according to Y. □

In the particular case in which the players are willing to accept a unique vector of weights λ, the scalar game associated with λ is called the λ-weighted game.

Definition 2.1. Let $(N, v) \in G^v$ and $\lambda \in \Lambda$. The λ-weighted game is the scalar game $(N, v_\lambda) \in g^v$, where N is the set of players whose characteristic function is given by $v_\lambda(S) = \lambda^t v(S)$ for every coalition $S \in \mathcal{N}$.

We denote by $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_p$ the extreme points of the polyhedron $\Lambda \subseteq \Lambda^m_{\geqq}$ and by Λ_E the matrix whose columns are the extreme points of Λ.

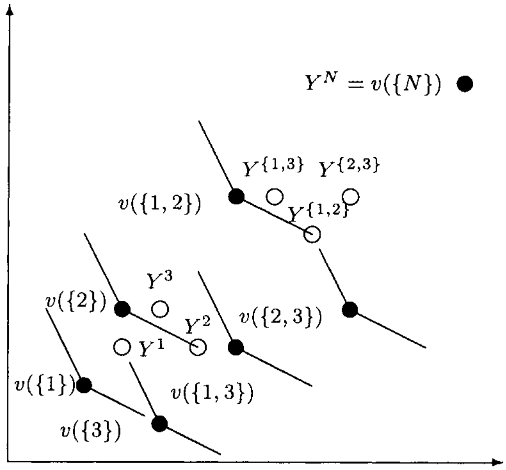

Fig. 1. Example 2.1 with $\Lambda = \{\lambda \in \Lambda^m_{\geqq} : 1/3 \leq \lambda_1 \leq 2/3\}$.

We call Λ-component games of the vector payoff game $(N, v) \in G^v$ to the $\bar{\lambda}_j$-weighted games, $(N, v_{\bar{\lambda}_j}) \in g^v, j = 1, 2, \ldots, p$. Notice that, when $\Lambda = \Lambda^m_{\geqq}$, there are m extreme points and the $\Lambda^m_{\geqq}$-component games are called component games. In this case, they are denoted by (N, v_j), $j = 1, 2, \ldots, m$, where

$$v_j : \mathcal{N} \cup \varnothing \to \mathbb{R}, \qquad v_j(S) = (v(S))_j, \qquad \forall j = 1, 2, \ldots, m.$$

Given a polyhedron of weights Λ, we say that X is an allocation of $(N, v) \in G^v$ if it verifies the efficiency property

$$\Lambda_E X^N = \Lambda_E v(N).$$

Besides, if Λ is a nondegenerated polyhedron, the allocations are matrices $X \in \mathbb{R}^{m \times n}$ verifying $X^N = v(N)$. The set of allocations of the game is denoted by $I^*(N, v)$.

Allocations verify only the efficiency property. Nevertheless, we should impose on them (according to the classical scalar theory) individual and collective rationality principles. The crucial point in the development of vector-valued games is the extension of the rationale from the scalar ordering to the ordering given in (1)–(2). To this end, we must replace the complete order $\leq$ in $\mathbb{R}$, for the comparison between side payments and the values of the characteristic function, by the ordering considered in $\mathbb{R}^m$. This very simple idea leads us to the following reformulation of the rationality principles and the domination through coalitions, depending on the ordering $\mathcal{R} \in \{\overset{\Lambda}{\gtrsim}, \overset{\Lambda}{\geqq}\}$:

(i) X verifies the individual rationality principle if $X^i \mathscr{R} v(\{i\})$ for any $i \in N$.

(ii) X verifies the collective rationality principle if $X^S \mathscr{R} v(S)$ for any $S \subseteq N$.

(iii) Let $X, Y \in \mathbb{R}^{m \times n}$, and let $S \in \mathscr{N}$ be a coalition. Y dominates X through S according to $\mathscr{R}$ and we will denote

$$Y \underset{\mathscr{R}}{\overset{S}{\mathrm{dom}}} X, \qquad \text{if } Y^S \overset{\Delta}{\geq} X^S \text{ and } v(S) \mathscr{R} Y^S,$$

$$\text{where } \overset{\Delta}{\geq} \text{ means } \overset{\Delta}{\geqq} \text{ and } \overset{\Delta}{\neq}.$$

These principles, when applied using the orderings given in (1)–(2), will lead us to different sets of stable allocations in vector-valued games, as we will see in the next sections.

3. Solution Concepts with Weak Ordering

The weakest ordering represents an attitude of compromise in the negotiation given by (1). The set of all the allocations is not acceptable by the players of a vector-valued game. An allocation is individually acceptable if it allocates to each player not less than the worth that the player can guarantee to himself. Notice that this concept depends on the meaning given to "not less". When the ordering is defined by (1), it leads us to the following definition of generalized imputation.

Definition 3.1. An allocation $X \in I^*(N, v)$ of the game $(N, v) \in G^v$ is a generalized imputation or simply an imputation if

$$X^i \overset{\Delta}{\gtrsim} v(\{i\}), \qquad \forall i \in N.$$

The set of all the imputations of the game will be denoted by $I(N, v; \overset{\Delta}{\gtrsim})$.

Now, we consider within the generalized imputation set the collective rationality principle. According to that principle, an imputation will be acceptable if no coalition can argue against its allocated amount X^S. To this end, we use the following dominance concept.

Definition 3.2. Let $X, Y \in \mathbb{R}^{m \times n}$, and let $S \in \mathscr{N}$ be a coalition. Y dominates X through S, according to $\overset{\Delta}{\gtrsim}$, and we will denote

$$Y \underset{\overset{\Delta}{\gtrsim}}{\overset{S}{\mathrm{dom}}} X, \qquad \text{if } Y^S \overset{\Delta}{\geq} X^S \text{ and } Y^S \overset{\Delta}{\lesssim} v(S).$$

In order to accomplish the above-mentioned rationality principles, we should consider those imputations which are not dominated with respect to any coalition. In doing that, no coalition has any incentive to complain against its allocation. Therefore, in this sense, this set of imputations is stable. The following definition states the concept of nondominated imputations.

Definition 3.3. An imputation $X \in I(N, v; \overset{\Delta}{\gtrsim})$ of the vector-valued game $(N, v) \in G^v$ is nondominated if, for any coalition $S \in \mathcal{N}$, there does not exist an imputation $Y \in I(N, v; \overset{\Delta}{\gtrsim})$ such that $Y \operatorname{dom}^S_{\overset{\Delta}{\gtrsim}} X$. The set of nondominated imputations is denoted by

$$\mathrm{NDI}(N, v; \overset{\Delta}{\gtrsim})$$
$$= \left\{ X \in I(N, v; \overset{\Delta}{\gtrsim}) : \nexists S \in \mathcal{N},\ Y \in I(N, v; \overset{\Delta}{\gtrsim}),\ Y \underset{\overset{\Delta}{\gtrsim}}{\overset{S}{\mathrm{dom}}} X \right\}.$$

In standard scalar cooperative games, the set of nondominated imputations has been considered widely; see Ref. 18 and the references therein. Nevertheless, in vector-valued cooperative games, the concept which plays an important role is the set of nondominated imputations by allocations, as it will be shown through the paper.

The same rationale of the idea of dominance is used in the definition if NDI can be strengthened looking for the dominant elements in the larger set of all the allocations. This idea leads us to a refinement of the set of nondominated imputations that we name the set of imputations nondominated by allocations (NDIA).

Definition 3.4. An imputation $X \in I(N, v; \overset{\Delta}{\gtrsim})$ of the vector-valued game $(N, v) \in G^v$ is nondominated by allocations if, for any coalition $S \in \mathcal{N}$, there does not exist an allocation $Y \in I^*(N, v)$ such that $Y \operatorname{dom}^S_{\overset{\Delta}{\gtrsim}} X$. This set is given by

$$\mathrm{NDIA}(N, v; \overset{\Delta}{\gtrsim})$$
$$= \{ X \in I(N, v; \overset{\Delta}{\gtrsim}) : \nexists S \in \mathcal{N},\ Y \in I^*(N, v),\ Y \underset{\overset{\Delta}{\gtrsim}}{\overset{S}{\mathrm{dom}}} X \}.$$

By their own definitions, it is clear that

$$\mathrm{NDIA}(N, v; \overset{\Delta}{\gtrsim}) \subseteq \mathrm{NDI}(N, v; \overset{\Delta}{\gtrsim}).$$

Although in general these two sets, $\mathrm{NDIA}(N, v; \overset{\Delta}{\gtrsim})$ and $\mathrm{NDI}(N, v; \overset{\Delta}{\gtrsim})$ are different, one can prove that, under mild conditions, they do coincide.

The following result proves that, by considering only the imputations in the set $\bar{I}(N, v; \overset{\Delta}{\gtrsim})$ whose elements verifies

$$X^i \overset{\Delta}{\gtrsim} v(\{i\}), \qquad X^i \neq v(\{i\}), \qquad \forall i \in N,$$

the following equality holds:

$$\mathrm{NDI}(N, v; \overset{\Delta}{\gtrsim}) = \mathrm{NDIA}(N, v; \overset{\Delta}{\gtrsim}).$$

Theorem 3.1. The following equality holds:

$$\mathrm{NDIA}(N, v; \overset{\Delta}{\gtrsim}) \cap \bar{I}(N, v; \overset{\Delta}{\gtrsim}) = \mathrm{NDI}(N, v; \overset{\Delta}{\gtrsim}) \cap \bar{I}(N, v; \overset{\Delta}{\gtrsim}).$$

Proof. For every game $(N, v) \in G^v$, we have

$$\mathrm{NDIA}(N, v; \overset{\Delta}{\gtrsim}) \cap \bar{I}(N, v; \overset{\Delta}{\gtrsim}) \subseteq \mathrm{NDI}(N, v; \overset{\Delta}{\gtrsim}) \cap \bar{I}(N, v; \overset{\Delta}{\gtrsim}).$$

Consider

$$X \in \mathrm{NDI}(N, v; \overset{\Delta}{\gtrsim}) \cap \bar{I}(N, v; \overset{\Delta}{\gtrsim}),$$

and assume that

$$X \notin \mathrm{NDIA}(N, v; \overset{\Delta}{\gtrsim}).$$

Thus,

$$\exists S \in \mathcal{N}, \ Y \in I^*(N, v) \text{ such that } Y \underset{\overset{\Delta}{\gtrsim}}{\overset{S}{\mathrm{dom}}} X,$$

that is,

$$Y^S \overset{\Delta}{\geqq} X^S \quad \text{and} \quad Y^S \overset{\Delta}{\lesssim} v(S).$$

Therefore,

$$X^S \overset{\Delta}{\leqq} Y^S \overset{\Delta}{\lesssim} v(S),$$

that is,

$$X^S \overset{\Delta}{\lesssim} v(S).$$

Then, if $-\Lambda^*$ denotes the polar set of $-\Lambda$, we have that

$$\exists d \in -\Lambda^*, \|d\| = 1, \exists \epsilon^0 > 0 \text{ such that}, \forall \epsilon < \epsilon^0, X^S + \epsilon d \overset{\Delta}{\lesssim} v(S). \tag{4}$$

As

$$X^i \overset{\Delta}{\gtrsim} v(\{i\}), \qquad X^i \overset{\Delta}{\ngeqq} v(\{i\}), \qquad \forall i \in N,$$

we have that

$$\exists \epsilon^1 > 0 \text{ such that } X^i - \epsilon d \overset{\Delta}{\gtrsim} v(\{i\}), \qquad \forall \epsilon < \epsilon^1, \forall i \in N.$$

Let

$$\epsilon \in (0, \min\{\epsilon^0, \epsilon^1\}),$$

and let

$$p \in N \backslash S.$$

Consider a matrix $Y \in \mathbb{R}^{m \times n}$ such that

$$Y^i = X^i + \epsilon d / |S|, \qquad \forall i \in S,$$

$$Y^p = X^p - \epsilon s d,$$

$$Y^i = X^i, \qquad \forall i \in N \backslash (S \cup \{p\}.)$$

The choice of ϵ means that

$$Y \in \bar{I}(N, v; \overset{\Lambda}{\gtrsim}) \subseteq I(N, v; \overset{\Lambda}{\gtrsim}).$$

Besides, since

$$Y^S = X^S + \epsilon d,$$

then by (4),

$$Y^S \overset{\Lambda}{\lesssim} v(S).$$

On the other hand,

$$Y^S - X^S = \epsilon d \overset{\Lambda}{\gtrsim} 0,$$

because

$$d \in -\Lambda^*.$$

Therefore,

$$Y \underset{\overset{\Lambda}{\gtrsim}}{\overset{S}{\text{dom}}} X.$$

But this is not possible because

$$X \in \text{NDI}(N, v; \overset{\Lambda}{\gtrsim}). \qquad \square$$

Therefore, nondominated imputations which do not belong to $\text{NDIA}(N, v; \overset{\Lambda}{\gtrsim})$ assign, to at least one of the players, a payoff that is equivalent to what he can obtain by himself.

Keeping track of the development followed in the standard theory, the next step is to impose the collective rationality to those imputations proposed as good allocations. This idea was suggested first in Ref. 19 and later formalized (for scalar games) under the name of core of the game.

Definition 3.5. The core of the vector-valued game $(N, v) \in G^v$ is defined as the set of allocations such that X^S is not dominated by $v(S)$ for every coalition S and is denoted by

$$C(N, v; \overset{\Lambda}{\gtrsim}) = \{X \in I^*(N, v): X^S \overset{\Lambda}{\gtrsim} v(S), \forall S \subset N\}.$$

The core above defined can be characterized alternatively by the following theorem.

Theorem 3.2. The following equality holds:

$$C(N, v; \overset{\Lambda}{\gtrsim})$$

$$= \{X \in I(N, v; \overset{\Lambda}{\gtrsim}): \nexists S \in \mathcal{N}, Y \in I^*(N, v), v(S) \overset{\Lambda}{\geqq} Y^S \overset{\Lambda}{\geq} X^S\}.$$

Proof. Consider

$$X \in C(N, v; \overset{\Lambda}{\gtrsim}).$$

Taking into account the individual coalitions,

$$X^i \overset{\Lambda}{\gtrsim} v(\{i\}), \qquad \forall i \in N,$$

and then

$$X \in I(N, v; \overset{\Lambda}{\gtrsim}).$$

Assume that there exists

$$S \in \mathcal{N}, \qquad Y \in I^*(N, v), \qquad v(S) \overset{\Lambda}{\geqq} Y^S \overset{\Lambda}{\geq} X^S.$$

That is to say

$$\lambda^t v(S) \geq \lambda^t X^S, \qquad \forall \lambda \in \Lambda,$$

$$\lambda^t v(S) > \lambda^t X^S, \qquad \forall \lambda \in \Lambda_{>}.$$

Then,

$$\nexists \lambda \in \Lambda \text{ such that } \lambda^t v(S) < \lambda^t X^S,$$

which contradicts

$$X \in C(N, v; \overset{\Lambda}{\gtrsim}).$$

Conversely, let

$$X \in I(N, v; \overset{\Lambda}{\gtrsim}),$$

and assume that a coalition S does not exist nor an allocation Y verifying

$$v(S) \overset{\Lambda}{\geqq} Y^S \overset{\Lambda}{\geq} X^S, \qquad S \subseteq N.$$

Then,

$$\nexists S \in \mathcal{N} \text{ such that } v(S) \overset{\Lambda}{\geq} X^S,$$

and there is no coalition $S \in \mathcal{N}$ satisfying

$$\lambda^t v(S) \geqq \lambda^t X^S, \qquad \forall \lambda \in \Lambda,$$

$$\lambda^t v(S) > \lambda^t X^S, \qquad \forall \lambda \in \Lambda_>.$$

Then,

$$\forall S \in \mathcal{N}, \exists \lambda \in \Lambda \text{ such that } \lambda^t v(S) < \lambda^t X^S$$

and

$$\exists \lambda' \in \Lambda_> \text{ such that } (\lambda')^t v(S) \leq (\lambda')^t X^S,$$

and therefore,

$$X^S \overset{\Lambda}{\gtrsim} v(S), \qquad \forall S \subset N. \qquad \square$$

A straightforward consequence of the above theorem that relates the core and the NDIA set is stated in the following corollary.

Corollary 3.1. The following relationship holds:

$$\text{NDIA}(N, v; \overset{\Lambda}{\gtrsim}) \subseteq C(N, v; \overset{\Lambda}{\gtrsim}).$$

We have followed the same rationale as in the standard theory of cooperative games when we have developed our solution concepts. Nevertheless, the behavior of the core in vector-valued cooperative games differs from the standard core in several respects. It is well known that essential, constant-sum games have always an empty core (see Ref. 20). Our next example shows that the same property is not true for vector-valued games.

Example 3.1. Consider a game with three players, $N = \{1, 2, 3\}$, two criteria, and payoff function given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}3\\4\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}8\\9\end{bmatrix}$	$\begin{bmatrix}9\\6\end{bmatrix}$	$\begin{bmatrix}10\\8\end{bmatrix}$	$\begin{bmatrix}12\\10\end{bmatrix}$

This is an essential constant-sum game, but the matrix

$$X = \begin{bmatrix} 1 & 10 & 1 \\ 5 & 1 & 4 \end{bmatrix}$$

is in the core. $\square$

A sufficient condition for the core of a game to be nonempty is that there exists $\hat{\lambda}$ in $\Lambda_>$ (the relative interior of Λ) such that the λ-weighted game $(N, v_{\hat{\lambda}}) \in g^v$ is balanced. Recall that a cooperative game is balanced if, for every nonnegative vector $(\alpha_S)_{S \subset N}$ satisfying

$$\sum_{\substack{S \\ i \in S \subset N}} \alpha_S = 1,$$

we have

$$\sum_{S \subset N} \alpha_S v(S) \leqq v(N).$$

Theorem 3.3. Consider the game $(N, v) \in G^v$, If $(N, \hat{\lambda}^t v)$ is balanced for some $\hat{\lambda} \in \Lambda_>$, then $C(N, v; \overset{\Lambda}{\gtrsim}) \neq \emptyset$.

Proof. If there exists $\hat{\lambda} \in \Lambda_>$ such that the λ-weighted game $(N, \hat{\lambda} v) \in g^v$ is balanced, it follows from the Bondareva–Shapley theorem that we may find $x \in C(N, v_{\hat{\lambda}})$. Consider $X \in \mathbb{R}^{m \times n}$, where

$$X^i = [x^i / \hat{\lambda}^t v(N)] v(N).$$

X is an allocation of the game $(N, v) \in G^v$, because

$$X^N = \sum_{i=1}^n X^i = \left[\sum_{i=1}^n x^i \Big/ \hat{\lambda}^t v(N) \right] v(N) = v(N).$$

Assume that

$$X \notin C(N, v; \overset{\Lambda}{\gtrsim});$$

that is,

$$\exists S \in \mathcal{N},\ Y \in I^*(N, v) \text{ such that } Y \operatorname{dom}^S_{\overset{\Lambda}{\geq}} X.$$

Then,

$$X^S \overset{\Lambda}{\lesssim} v(S),$$

and therefore,

$$\lambda^t X^S < \lambda^t v(S), \qquad \forall \lambda \in \Lambda_>,$$

$$\lambda^t X^S \leq \lambda^t v(S), \qquad \forall \lambda \in \Lambda.$$

As $\hat{\lambda} \in \Lambda_>$, then

$$\hat{\lambda}^t X^S < \hat{\lambda}^t v(S),$$

but

$$\begin{aligned}\hat{\lambda}^t X^S &= \sum_{i\in S} \hat{\lambda}^t X^i \\ &= \left[\sum_{i\in S} x^i \Big/ \hat{\lambda}^t v(N)\right] \hat{\lambda}^t v(N) \\ &= x^S \\ &\geq \hat{\lambda}^t v(S) \\ &= v_{\hat{\lambda}}(S), \qquad \forall S \in \mathcal{N},\end{aligned}$$

and this is not possible. □

The converse is not true as shown in the next example.

Example 3.2. Consider the game $(N, v) \in G^v$ with four players, $N = \{1, 2, 3, 4\}$, two criteria, and payoff function given by

S	$\{1\}, \{2\}$ $\{3\}, \{4\}$	$\{1, 2\}, \{3, 4\}, \{1, 2, 3\}$ $\{1, 2, 4\}, \{1, 3, 4\}$	$\{1, 3\}, \{2, 4\}$	$\{1, 4\}, \{2, 3\}$	$\{2, 3, 4\}$	$\{1, 2, 3, 4\}$
$v(S)$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}0\\3\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}3\\3\end{bmatrix}$.

The allocation X verifies

$$X = \begin{bmatrix} -1 & 1 & 2 & 1 \\ 2 & 1 & -1 & 1 \end{bmatrix} \in C(N, v; \overset{\wedge}{\gtrsim}),$$

because

$$X^S \not\leq v(S), \qquad \forall S \in \mathcal{N}.$$

However, there does not exist

$$w = (\lambda, 1 - \lambda) \in \Lambda^2_{\geqq}$$

such that the corresponding scalar weighted game is balanced. Indeed, the characteristic function for this scalar game is given below:

S	$\{1\}, \{2\}, \{3\}, \{4\}$ $\{1, 3\}, \{2, 4\}$	$\{1, 2\}, \{3, 4\}$ $\{1, 2, 3\}$ $\{1, 2, 4\}, \{1, 3, 4\}$	$\{1, 4\}, \{2, 3\}$	$\{2, 3, 4\}$	N
$\lambda^t v(S)$	1	2	$3 - 3\alpha$	$1 + 3\alpha$	3 ,

and if we consider the balanced collection

$$\beta = \{\{1\}, \{2\}, \{3\}, \{4\}\},$$

with balancing weights

$$\alpha_S = 1, \qquad \forall S \in \beta,$$

it follows that

$$\sum_{S \in \beta} \alpha_S w^t v(S) = 4 \not\leqq 3 = v(N). \qquad \square$$

4. Solution Concepts with Strong Ordering

So far, we have analyzed solution concepts for vector-valued cooperative games assuming that the players agree with the ordering (1). This ordering implies that the coalitions admit allocations which are not worse in all the criteria. Nevertheless, it may happen that, in some situations, the preference structure assumed by the agents is stronger, and the coalitions accept only allocations if they get more than the worth given by the characteristic function. This assumption modifies the rationale of the decision process under the game and, therefore, the solution concepts will be modified accordingly.

In this section, we analyze the concepts of nondominated imputations, nondominated imputations by allocations, and their relationships with the core when the ordering (2) is assumed by the agents in the game.

Definition 4.1. Let $\Lambda \subseteq \Lambda^m_{\geqq}$. An allocation $X \in I^*(N, v)$ of the game $(N, v) \in G^v$ is a preference imputation if

$$X^i \overset{\Lambda}{\geqq} v(\{i\}), \qquad \forall i \in N.$$

The set of all preference imputations in the game $(N, v) \in G^v$ will be denoted by $I(N, v; \overset{\Lambda}{\geqq})$.

It is worth noting that

$$I(N, v; \overset{\Lambda}{\geqq}) \subseteq I(N, v; \overset{\Lambda}{\gtrsim}).$$

This is a straightforward consequence of their definitions.

Definition 4.2. Let $X, Y \in \mathbb{R}^{m \times n}$, and let $S \in \mathcal{N}$ be a coalition. Y dominates individually X through S according to $\overset{\Lambda}{\geq}$, and we will denote

$$Y \underset{\overset{\Lambda}{\geq}}{\overset{S}{\text{domi}}} X, \qquad \text{if } Y^i \overset{\Lambda}{\geq} X^i, \forall i \in S \quad \text{and} \quad Y^S \overset{\Lambda}{\leqq} v(S).$$

Y dominates X through S according to $\stackrel{\Delta}{\geq}$, and we will denote

$$Y \operatorname{dom}^{S}_{\stackrel{\Delta}{\geq}} X, \qquad \text{if } Y^S \stackrel{\Delta}{\geq} X^S \quad \text{and} \quad Y^S \stackrel{\Delta}{\leqq} v(S).$$

Theorem 4.1. Let X be an allocation of the game $(N, v) \in G^v$. The following statements are equivalent:

(i) $\exists\, Y \in I^*(N, v)$ such that $Y \operatorname{domi}^{S}_{\stackrel{\Delta}{\geq}} X$.

(ii) $\exists\, Y \in I^*(N, v)$ such that $Y \operatorname{domi}^{S}_{\stackrel{\Delta}{\geq}} X$.

(iii) $X^S \stackrel{\Delta}{\leq} v(S)$.

Proof.

(i) ⇒ (ii). If $\exists\, Y \in I^*(N, v)$ such that $Y \operatorname{domi}^{S}_{\stackrel{\Delta}{\gtrsim}} X$. then

$$Y^i \stackrel{\Delta}{\geq} X^i, \qquad \forall i \in S \quad \text{and} \quad Y^S \stackrel{\Delta}{\leqq} v(S).$$

Then

$$Y^S \stackrel{\Delta}{>} X^S \quad \text{and} \quad Y^S \stackrel{\Delta}{<} v(S),$$

that is,

$$Y \operatorname{dom}^{S}_{\stackrel{\Delta}{\leq}} X.$$

(ii) ⇒ (iii). As $X^S \stackrel{\Delta}{\leq} Y^S \stackrel{\Delta}{\leqq} v(S)$,

$$X^S \stackrel{\Delta}{\leq} v(S).$$

(iii) ⇒ (i). Let

$$d = v(S) - X^S \stackrel{\Delta}{\geq} 0.$$

It is possible to construct a matrix $Y \in \mathbb{R}^{m \times n}$ with

$$Y^i = X^i + d/|S|, \qquad \forall i \in S,$$

$$Y^i = X^i - d/|-S|, \quad \forall i \notin S.$$

The matrix Y is a preimputation of the game because

$$Y^N = X^N = v(N).$$

Moreover,

$$Y^i \stackrel{\Delta}{\geq} X^i, \qquad \forall i \in S,$$

and

$$Y^S = X^S + d = v(S).$$

So,

$$Y \operatorname*{domi}^{S}_{\overset{\Lambda}{\geq}} X. \qquad \square$$

It is straightforward to see that items (i) and (ii) in the thesis of this result are also equivalent in the set of imputations. This is to say that the dominance through a coalition S is equivalent to the dominance for all the players in that coalition.

Once more, we use the concept of dominance through coalitions, given in Definition 4.2, to introduce the set of nondominated preference imputations.

Definition 4.3. A preference imputation $X \in I(N, v; \overset{\Lambda}{\geqq})$ in the game $(N, v) \in G^v$ is said to be nondominated if no coalition $S \in \mathcal{N}$ can find another imputation $Y \in I(N, v; \overset{\Lambda}{\geqq})$ such that $Y \operatorname{dom}^S_{\overset{\Lambda}{\geq}} X$.

$$\begin{aligned} &\mathrm{NDI}(N, v; \overset{\Lambda}{\geqq}) \\ &= \left\{ X \in I(N, v; \overset{\Lambda}{\geqq}) : \nexists S \in \mathcal{N},\ Y \in I(N, v; \overset{\Lambda}{\geqq}),\ Y \operatorname*{dom}^{S}_{\overset{\Lambda}{\geq}} X \right\}. \end{aligned}$$

In the preference imputation set, we may find nondominated preference imputations using nondominated imputations of the scalar Λ-components games.

Recall that we denote by $\Lambda_E \in \mathbb{R}^{m \times p}$ the matrix of the extreme points of the polyhedron of weights Λ.

Theorem 4.2. If the rows of $\Lambda_E^t X \in \mathbb{R}^{m \times p}$ are nondominated imputations of the scalar Λ-component games of $(N, v) \in G^v$, then $X \in \mathrm{NDI}(N, v; \overset{\Lambda}{\geqq})$.

Proof. Suppose that

$$X \notin \mathrm{NDI}(N, v; \overset{\Lambda}{\geqq}).$$

Then,

$$\exists S \in \mathcal{N},\ Y \in (N, v; \overset{\Lambda}{\geqq}) \text{ such that } Y \operatorname*{dom}^{S}_{\overset{\Lambda}{\geq}} X,$$

that is,

$$Y^i \overset{\Lambda}{\geq} X^i, \qquad \forall i \in S \quad \text{and} \quad Y^S \overset{\Lambda}{\leqq} v(S).$$

Therefore,

$$\lambda^t Y^i > \lambda^t X^i, \qquad \forall i \in S, \qquad \forall \lambda \in \Lambda_>,$$

$$\lambda^t Y^S \leq \lambda^t v(S), \qquad \forall \lambda \in \Lambda.$$

As

$$\lambda = \Lambda_E \alpha, \, \alpha \in \Lambda^p_{\geqq}, \qquad \forall \lambda \in \Lambda_>,$$

we have

$$\alpha^t \Lambda^t_E Y^i > \alpha^t \Lambda^t_E X^i, \qquad \forall i \in S, \forall \alpha \in \Lambda^p_{\geqq},$$

$$\alpha^t \Lambda^t_E Y^S \leq \alpha^t \Lambda^t_E v(S), \quad \forall \alpha \in \Lambda^p_{\geqq}.$$

Then,

$$\Lambda^t_E Y^i \geq \Lambda^t_E X^i, \qquad \forall i \in S,$$

$$\Lambda^t_E Y^S \leqq \Lambda^t_E v(S).$$

Hence, for every scalar Λ-component game,

$$\bar{\lambda}^t_j Y^i \geq \bar{\lambda}^t_j X^i, \qquad \forall i \in S,$$

$$\bar{\lambda}^t_j Y^S = (\bar{\lambda}^t_j Y)^S \leqq \bar{\lambda}^t_j v(S),$$

contradicting that $\bar{\lambda}^t_j X$ are nondominated imputations in the scalar Λ-component games $(N, v_{\bar{\lambda}_j})$. □

The converse is not always true; that is, there exist nondominated preference imputations in the game $(N, v) \in G^v$ apart from those obtained from nondominated imputations of each scalar Λ-component game. The following example shows a game with this property.

Example 4.1. Consider a game $(N,v) \in G^v$, with three players, $N = \{1, 2, 3\}$, two criteria, $\Lambda = \mathbb{R}^2_{\geqq}$, and payoff function given by

S	$\{1\}, \{2\}, \{3\}$	$\{1, 2\}$	$\{1, 3\}, \{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

The imputation

$$X = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

is nondominated. However, the allocation given to the second component game $X_2 = (1, 1, 2)$ is dominated for the coalition $S = \{1, 2\}$ by

$$y = (3/2, 3/2, 1).$$

Therefore,

$$X_2 \notin \mathrm{NDI}(N, v_2). \qquad \square$$

This example is very important. It shows that the analysis of vector-valued games, even with the strong ordering (2), does not reduce to the analysis of its Λ-component games. Therefore, although in some cases it may seem that the componentwise analysis is enough, it does not extend to all the concepts of solution.

In the preference imputation set, there may be nondominated imputations which may be dominated provided that we enlarge the set of admissible allocations. This is shown in the next example.

Example 4.2. Consider the game $(N, v) \in G^v$, with three players, $N = \{1, 2, 3\}$, two criteria, $\Lambda = \mathbb{R}^2_{\geqq}$, and payoff function given by

S	$\{1\}, \{2\}, \{3\}$	$\{1, 2\}$	$\{1, 3\}, \{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

The imputation

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \in \mathrm{NDI}(N, v; \overset{\Lambda}{\geqq}).$$

But the players in coalition $S = \{1, 2\}$ can improve their payments with respect to the first criterion through an allocation that is not an imputation. However, the matrix

$$Y = \begin{bmatrix} 1 & 1 & 1 \\ 3/2 & 3/2 & 2 \end{bmatrix}$$

is a nondominated imputation, too. But in this case, the allocation is stabler than the imputation X, because no coalition can improve for itself.

This example leads us to redefine the nondominated imputation set to a smaller set of preference imputations that we call nondominated by allocations. This concept is analogous to the one introduced in the previous section, except with regard to the ordering given by (2). Therefore, we proceed similarly as in Section 3.

Definition 4.4. A preference imputation $X \in I(N, v; \overset{\vartriangle}{\geqq})$ in the game $(N, v) \in G^v$ is nondominated by allocations if no coalition $S \in \mathcal{N}$ can find another allocation Y such that $Y \operatorname{dom}^S_{\overset{\vartriangle}{\gtrsim}} X$.

$$\mathrm{NDIA}(N, v; \overset{\vartriangle}{\geqq})$$
$$= \left\{ X \in I(N, v; \overset{\vartriangle}{\geqq}) \colon \nexists S \in \mathcal{N},\ Y \in I^*(N, v) \colon Y \underset{\overset{\vartriangle}{\geqq}}{\overset{S}{\operatorname{dom}}} X \right\}.$$

Notice that

$$\mathrm{NDIA}(N, v; \overset{\vartriangle}{\geqq}) \subseteq \mathrm{NDI}(N, v; \overset{\vartriangle}{\geqq}).$$

In general, these two sets are different, but they do coincide under mild conditions such as zero monotonicity.

Proceeding similarly as in Section 3, we introduce now the concept of core with respect to the ordering (2).

Definition 4.5. The preference core of a cooperative vector-valued game $(N, v) \in G^v$ is the set of allocations $X \in I^*(N, v)$ such that

$$X^S \overset{\vartriangle}{\geqq} v(S), \qquad \forall S \subset N.$$

We will denote this set as $C(N, v; \overset{\vartriangle}{\geqq})$.

Our first result proves that this core coincides with the set $\mathrm{NDIA}(N, v; \overset{\vartriangle}{\gtrsim})$.

Theorem 4.3. The following identity holds:

$$C(N, v; \overset{\vartriangle}{\geqq})$$
$$= \{ X \in I(N, v; \overset{\vartriangle}{\gtrsim}) \colon \nexists S \subseteq \mathcal{N},\ Y \in I^*(N, v), v(S) \overset{\vartriangle}{\gtrsim} Y^S \overset{\vartriangle}{\geqq} X^S \}.$$

Proof. Suppose that

$$X \in C(N, v; \overset{\vartriangle}{\geqq})$$

and

$$\exists S \in \mathcal{N} \text{ and } Y \in I^*(N, v) \text{ such that } v(S) \overset{\vartriangle}{\gtrsim} Y^S \geq X^S.$$

Then, it is not possible that

$$X^S \overset{\Delta}{\geqq} v(S) \text{ and } X \notin C(N, v; \overset{\Delta}{\geqq}).$$

Reciprocally, suppose that

$$X \in I(N, v; \overset{\Delta}{\gtrsim})$$

and

$$\nexists S \in \mathcal{N},\ Y \in I^*(N, v) \text{ such that } v(S) \overset{\Delta}{\gtrsim} Y^S \overset{\Delta}{\geq} X^S.$$

If

$$X \notin C(N, v; \overset{\Delta}{\geqq}),$$

we have, that

$$\exists S \in \mathcal{N} \text{ verifying } v(S) \overset{\Delta}{\gtrsim} X^S \quad \text{and} \quad v(S) \overset{\Delta}{\ngeqq} X^S.$$

Then, we can consider the allocation Y such that

$$v(S) \overset{\Delta}{\gtrsim} Y^S \overset{\Delta}{\geq} X^S. \qquad \square$$

Next, we establish that a necessary and sufficient condition for the existence of imputations in the preference core is balancedness; that is, for every nonnegative vector $(\alpha_S)_{S \subset N}$ satisfying

$$\sum_{\substack{S \\ i \in S \subset N}} \alpha_S = 1,$$

we have

$$\sum_{S \subset N} \alpha_S v(S) \leqq v(N).$$

Consider the nondegenerated polyhedron $\Lambda \in \Lambda^m_{\geqq}$, whose extreme points are $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_p$. Let $\Lambda_E \in \mathbb{R}^{m \times p}$ be the matrix whose columns are the extreme points of Λ.

Theorem 4.4. A necessary and sufficient condition for $C(N, v; \overset{\Delta}{\geqq})$ being nonempty is that the p scalar weighted games $(N, v_{\bar{\lambda}_j}) \in g^v, j = 1, 2, \ldots, p$, are balanced.

Proof. If $C(N, v; \overset{\Delta}{\geqq}) \neq \emptyset$, let

$$X \in C(N, v; \overset{\Delta}{\geqq}).$$

Then,

$$X^N = v(N) \quad \text{and} \quad X^S \overset{\Delta}{\geqq} v(S), \qquad \forall S \in \mathcal{N};$$

that is,

$$\lambda^t X^S \geq \lambda^t v(S), \qquad \forall S \in \mathcal{N}, \forall \lambda \in \Lambda.$$

For the extreme points of Λ,

$$\bar{\lambda}_j^t X^S \geq \bar{\lambda}_j^t v(S), \qquad \forall S \in \mathcal{N}, \forall j = 1, 2, \dots, p;$$

that is to say,

$$(\bar{\lambda}_j^t X)^S \geq (\bar{\lambda}_j^t v)(S), \qquad \forall S \in \mathcal{N}, \forall j = 1, 2, \dots, p.$$

As

$$\begin{aligned}(\bar{\lambda}_j^t X)^N &= \bar{\lambda}_j^t X^N \\ &= \bar{\lambda}_j^t v(N),\end{aligned}$$

it holds that

$$\bar{\lambda}_j^t X \in C(N, v_{\bar{\lambda}_j}), \qquad \forall j = 1, 2, \dots, p,$$

and the games $(N, v_{\bar{\lambda}_j})$ are balanced.

Conversely, as the scalar Λ-component games are balanced, there exist elements in each scalar weighted game core. Consider the matrix $B \in \mathbb{R}^{pxn}$ whose rows are imputations in the core of the corresponding Λ-component game. It is easy to see that

$$B = \Lambda_E^t X,$$

where X is an allocation of $(N, v) \in G^v$. Assume that

$$X \notin C(N, v; \overset{\Lambda}{\geqq}).$$

Then,

$$\exists S \in \mathcal{N}, \lambda \in \Lambda \text{ such that } \lambda^t X^S < \lambda^t v(S);$$

that is,

$$\exists S \in \mathcal{N}, \alpha \in \Lambda^p_{\geqq} \text{ such that } \alpha^t \Lambda_E^t X^S < \alpha^t \Lambda_E^t v(S),$$

and therefore,

$$\Lambda_E^t X^S = (\Lambda_E^t X)^S \ngeqq \Lambda_E^t v(S).$$

Then,

$$\exists j \in \{1, 2, \dots, p\}, S \in \mathcal{N} \text{ such that } \bar{\lambda}_j^t X^S < \bar{\lambda}_j^t v(S),$$

contradicting

$$\bar{\lambda}^t X \in C(N, v_{\bar{\lambda}_j}). \qquad \square$$

This result states that, considering the preference imputations, the concept of core reduces to the Cartesian product of the cores of the weighted Λ-component games. Thus, this core may be empty (as soon as one of the component games has empty core, see e.g. Example 1.1), but this cannot be applied to the set of preference imputations nondominated by allocations. This reason gives an important role to this set as solution concept.

Next, we are going to show that there are vector-valued games in which the core is empty but the set of preference imputations nondominated by allocations is nonempty.

Example 4.3. Consider the game $(N, v) \in G^v$, where $N = \{1, 2, 3\}$, there are two criteria, $\Lambda = \mathbb{R}^2_{\geqq}$, and the characteristic function is given by

S	$\{1\}, \{2\}, \{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}3\\5\end{bmatrix}$	$\begin{bmatrix}4\\4\end{bmatrix}$	$\begin{bmatrix}3\\5\end{bmatrix}$	$\begin{bmatrix}6\\6\end{bmatrix}$

The second component of this game is nonbalanced because, for the balanced collection,

$$\beta = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\},$$

with balanced coefficients

$$\alpha_S = 1/2, \qquad \forall S \in \beta,$$

it holds that

$$(1/2)v(\{1, 2\}) + (1/2)v(\{1, 3\}) + (1/2)v(\{2, 3\}) = (5, 7)^t \nleqq v(N).$$

Therefore, the preference core is empty. However, the matrix

$$X = \begin{bmatrix} 2 & 3/2 & 5/2 \\ 2 & 2 & 2 \end{bmatrix}$$

is an imputation of the game nondominated by allocations. □

Our next result shows that this core is always included in the set of preference imputations nondominated by allocations.

Theorem 4.5. The following relationship holds:

$$C(N, v; \overset{\Lambda}{\geqq}) \subseteq \mathrm{NDIA}(N, v; \overset{\Lambda}{\geqq}).$$

Proof. Let X be an element in the core,

$$X \in C(N, v; \overset{\Lambda}{\geqq}).$$

Then,

$$X^S \overset{\Lambda}{\geqq} v(S), \qquad \forall S \subseteq N \quad \text{and} \quad X^N = v(N).$$

Assume that

$$\exists S \in \mathcal{N},\ Y \in I^*(N, v)\colon\ Y \underset{\overset{\Lambda}{\geqq}}{\overset{S}{\mathrm{dom}}} X.$$

Then,

$$X^S \overset{\Lambda}{\leqq} v(S),$$

contradicting

$$X \in C(N, v; \overset{\Lambda}{\geqq}).$$

So,

$$X \in \mathrm{NDIA}(N, v; \overset{\Lambda}{\geqq}).$$ □

The next example, in which $\Lambda = \Lambda^2_{\geqq}$, shows that the inclusion may be strict.

Example 4.4. Consider the game $(N, v) \in G^v$, with three players, $N = \{1, 2, 3\}$, two criteria, and characteristic function defined by

S	$\{1\}, \{2\}, \{3\}$	$\{1, 2\}$	$\{1, 3\}, \{2, 3\}$	$\{1, 2, 3\}$
$v(S)$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}3\\2\end{bmatrix}$	$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}5\\4\end{bmatrix}$

The matrix

$$X_1 = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

is an imputation in the Cartesian product core of the game and then an imputation nondominated by allocations; however, the matrix

$$X_2 = \begin{bmatrix} 1 & 3/2 & 5/2 \\ 3/2 & 3/2 & 1 \end{bmatrix}$$

is an imputation nondominated by allocations, which is not in the Cartesian product core, because

$$X_{\{1,2\}} \not\geqq v(\{1,2\}). \qquad \square$$

Finally, to conclude this section we establish the relationship that exists between the set of preference imputations nondominated by allocations and the core defined with respect to the ordering (1). We prove that this set coincides with the core $C(N, v; \overset{\vartriangle}{\gtrsim})$ provided that we consider only preference imputations.

Theorem 4.6. The following relationship holds:

$$\mathrm{NDIA}(N, v; \overset{\vartriangle}{\geqq}) = C(N, v; \overset{\vartriangle}{\gtrsim}) \cap I(N, v; \overset{\vartriangle}{\geqq})$$

Proof. First, consider

$$X \in I(N, v; \overset{\vartriangle}{\geqq})$$

such that

$$X \in C(N, v; \overset{\vartriangle}{\gtrsim}) = \{X \in I^*(N, v): X^S \overset{\vartriangle}{\gtrsim} v(S), \forall S \subset N\}.$$

Suppose that

$$X \notin \mathrm{NDIA}(N, v; \overset{\vartriangle}{\gtrsim}).$$

Then,

$$\exists S \in \mathcal{N},\ Y \in I^*(N, v) \text{ such that } v(S) \overset{\vartriangle}{\geqq} Y^S \overset{\vartriangle}{\geq} X^S.$$

Reciprocally, let X be in $\mathrm{NDIA}((N, v; \overset{\vartriangle}{\geqq})$. Then,

$$\nexists S \in \mathcal{N} \text{ such that } v(S) \overset{\vartriangle}{\geq} X^S.$$

Therefore,

$$X^S \overset{\vartriangle}{\gtrsim} v(S), \qquad \forall S \subset N. \qquad \square$$

The NDIA solution concept plays a crucial role in the theory of vector-valued games, in that it is a valid solution concept for both orderings. On the one hand, the preference core $C(N, v; \overset{\vartriangle}{\geqq})$ coincides with the set $\mathrm{NDIA}(N, v; \overset{\vartriangle}{\gtrsim})$ with respect to the ordering (1) as stated in Theorem 4.3. On the other hand, the set $\mathrm{NDIA}(N, v; \overset{\vartriangle}{\geqq})$ with respect to the ordering (2) reduces to $C(N, v; \overset{\vartriangle}{\gtrsim})$ on the set of preference imputations as stated in Theorem 4.6.

A summary of the relationships that hold between the different sets considered in the paper is shown in Fig. 2.

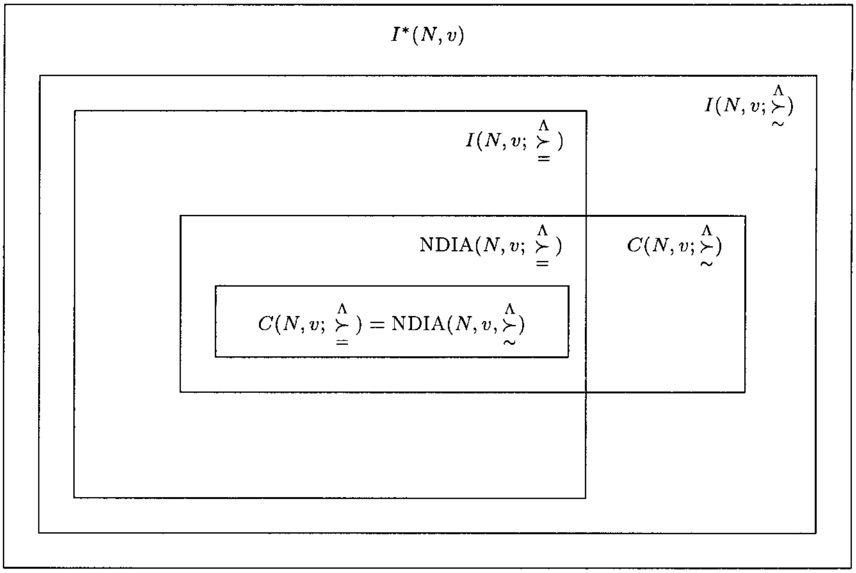

Fig. 2. Relationships between the different solution concepts.

5. Conclusions

The analysis of vector-valued games depends on the ordering assumed to compare side payments and payoffs. We have introduced two different domination relationships. In the first one, the players agree on side payments so that they do not get less in all the components of their payoff functions. Therefore, it represents an attitude toward compromise in the negotiation. In the second one, players accept side payments which at least ensure what they can guarantee by themselves. Thus, it represents an attitude toward independent negotiation.

These two domination relationships lead to two different theories of vector-valued games with transferable utility (TU games) whose solution concepts are different. These two theories allow us to define the sets of nondominated imputations by allocations. In the paper, we show that the set of nondominated imputations by allocations with respect to a given domination structure characterizes the core with respect to the other domination relation. This is shown in Theorems 4.3 and 4.6.

In conclusion, the sets NDIA play a crucial role as a solution concept in vector-valued cooperative games because they can be recommended as solution according to the attitude toward the negotiation exhibited by the players in the game.

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